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B. Sc. (Honrs) Part 1 Paper 2

Subject Mathematics

Title/Heading of topic : Equivalence Relation

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Equivalence Realation

Yoʊʳᵈˡˡ vɯ from your early stɑʊʲy of fractions that each fraction has many equivalent forms. For example,

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{-1}{-2}, \frac{-3}{-6}, \frac{15}{30}, \dots$$

are all different ways to represent the same number. They may look different; they may be called different names; but they are all equal. The idea of grouping together things that "look different but are really the same" is the central idea of equivalence relations.

A partition of a set S is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is S.

Definition 1. *A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets $A_i, i \in I$ (where I is an index set) forms a partition of S if and only if*

(i) $A_i \neq \emptyset$ *for $i \in I$,*

(ii) $A_i \cap A_j = \emptyset$ *when $i \neq j$, and*

(iii)

$$\bigcup_{i \in I} A_i = S.$$

(Here the notation $\bigcup_{i \in I}$ represents the union of the sets A_i for all $i \in I$.)

Definition 2. *A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.*

Recall:

1. R is reflexive if, and only if, $\forall x \in A, x\,R\,x$.

2. R is symmetric if, and only if, $\forall x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.

3. R is transitive if, and only if, $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

Definition 3. *Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.*

Example 1. *Are these equivalence relations on $\{0, 1, 2\}$?*

(a) $\{(0,0), (1,1), (0,1), (1,0)\}$

(b) $\{(0,0), (1,1), (2,2), (0,1), (1,2)\}$

(c) $\{(0,0), (1,1), (2,2), (0,1), (1,2), (1,0), (2,1)\}$

(d) $\{(0,0), (1,1), (2,2), (0,1), (0,2), (1,0), (1,2), (2,0), (2,1)\}$

(e) $\{(0,0), (1,1), (2,2)\}$

Solution. (a) R is not reflexive: $(2,2) \notin R$. Thus, by definition, R is not an equivalence relation.

(b) R is not symmetric: $(1,2) \in R$ but $(2,1) \notin R$. Thus R is not an equivalence relation.

(c) R is not transitive: $(0,1), (1,2) \in R$, but $(0,2) \notin R$. Thus R is not an equivalence relation.

(d) R is reflexive, symmetric, and transitive. Thus R is an equivalence relation.

(e) R is reflexive, symmetric, and transitive. Thus R is an equivalence relation. $\square$

Example 2. *Which of these relations on the set of all functions on $\mathbb{Z} \to \mathbb{Z}$ are equivalence relations?*

(a) $R = \{(f,g) \mid f(1) = g(1)\}$.

(b) $R = \{(f,g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$.

Solution. (a) $f(1) = f(1)$, so R is reflexive. If $f(1) = g(1)$, then $g(1) = f(1)$, so R is symmetric. If $f(1) = g(1)$ and $g(1) = h(1)$, then $f(1) = h(1)$, so R is transitive. R is reflexive, symmetric, and transitive, thus R is an equivalence relation.

(b) $f(1) = f(1)$, so R is reflexive. If $f(1) = g(1)$ or $f(0) = g(0)$, then $g(1) = f(1)$ or $g(0) = f(0)$, so R is symmetric. However, R is not transitive: if $f(0) = g(0)$ and $g(1) = h(1)$, it does not necessarily follow that $f(1) = h(1)$ or that $f(0) = h(0)$. Thus R is not an equivalence relation. $\square$

Example 3. *Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ such that*

$$((a,b),(c,d)) \in R \Leftrightarrow a+d = b+c.$$

Show that R is an equivalence relation.

Solution. R *is reflexive:* Suppose (a,b) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$. [We must show that $(a,b) R (a,b)$.] We have $a+b = a+b$. Thus, by definition of R, $(a,b) R (a,b)$.

R *is symmetric:* Suppose (a,b) and (c,d) are two ordered pairs in $\mathbb{Z} \times \mathbb{Z}$ and $(a,b) R (c,d)$. [We must show that $(c,d) R (a,b)$.] Since $(a,b) R (c,d)$, $a+d = b+c$. But this implies that $b+c = a+d$, and so, by definition of R, $(c,d) R (a,b)$.

R *is transitive:* Suppose $(a,b), (c,d)$, and (e,f) are elements of $\mathbb{Z} \times \mathbb{Z}$, $(a,b) R (c,d)$, and $(c,d) R (e,f)$. [We must show that $(a,b) R (e,f)$.] Since $(a,b) R (c,d)$, $a+d = b+c$, which means $a-b = c-d$, and since $(c,d) R (e,f)C$, $c+f = d+e$, which means $c-d = e-f$. Thus $a-b = e-f$, which means $a+f = b+e$, and so, by definition of R, $(a,b) R (e,f)$. $\square$

Definition 4. *Suppose A is a set and R is an equivalence relation on A. For each element a in A, the equivalence class of a, denoted $[a]$ and called the class of a for short, is the set of all elements x in A such that x is related to a by R.*

In symbols,

$$[a] = \{x \in A \mid x\,R\,a\}.$$

The procedural version of this definition is

$$\forall x \in A, \quad x \in [a] \Leftrightarrow x\,R\,a.$$

When several equivalence relations on a set are under discussion, the notation $[a]_R$ is often used to denote the equivalence class of a under R.

Theorem 1. *Let R be an equivalence relation on a set A. Let $a,b \in A$. The following are equivalent (TFAE):*

(i) $a\,R\,b$

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$.

Proof. [(i) $\Rightarrow$ (ii)]: Assume that $a\,R\,b$. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then $a\,R\,c$. Because $a\,R\,b$ and R is symmetric, we know that $b\,R\,a$. Furthermore, because R is transitive and $b\,R\,a$ and $a\,R\,c$, it follows that $b\,R\,c$. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar.

[(ii) $\Rightarrow$ (iii)]: Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

[(iii) $\Rightarrow$ (i)]: Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, $a\,R\,c$ and $b\,R\,c$. By the symmetric property, $c\,R\,b$. Then by transitivity, because $a\,R\,c$ and $c\,R\,b$, we have $a\,R\,b$.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent. $\qquad\square$

Corollary. *If A is a set, R is an equivalence relation on A, and a and b are elements of A, then*

$$\text{either } [a] \cap [b] = \emptyset \text{ or } [a] = [b].$$

That is, any two equivalence classes of an equivalence relation are either mutually disjoint or identical.

Theorem 2. *Let R be an equivalence relation on a set A. Then the equivalence classes of R form a partition of A. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set A, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.*

The proof of Theorem 2 is divided into two parts: first, a proof that A is the union of the equivalence classes of R and second, a proof that the intersection of any two distinct equivalence classes is empty. The proof of the first part follows from the fact that the relation is reflexive. The proof of the second part follows from the corollary above.

Proof. Suppose A is a set and R is an equivalence relation on A. For notational simplicity, we assume that R has only a finite number of distinct equivalence classes, which we denote

$$A_1, A_2, \ldots, A_n,$$

where n is a positive integer. (When the number of classes is infinite, the proof is identical except for notation.)

$(A = A_1 \cup A_2 \cup \cdots \cup A_n)$: [We must show that $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq A$.]

To show that $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$, suppose x is an arbitrary element of A. [We must show that $x \in A_1 \cup A_2 \cup \cdots \cup A_n$.] By reflexivity of R, $x\,R\,x$. But this implies that $x \in [x]$ by definition of class. Since x is in some equivalence class, it must be in one of the distinct equivalence classes $A_1, A_2, \ldots$, or A_n. Thus $x \in A_i$ for some index i, and hence $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ by definition of union [as was to be shown].

To show that $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq A$, suppose $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. [We must show that $x \in A$.] Then $x \in A_i$ for some $i = 1, 2, \ldots, n$, by definition of union. But each A_i is an equivalence class of R. And equivalence classes are subsets of A. Hence $A_i \subseteq A$ and so $x \in A$ [as was to be shown].

Since $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq A$, then by definition of set equality, $A = A_1 \cup A_2 \cup \cdots \cup A_n$.

(The distinct classes of R are mutually disjoint): Suppose that A_i and A_j are any two distinct equivalence classes of R. [We must show that A_i and A_j are disjoint.] Since A_i and A_j are distinct, then $A_i \neq A_j$. And since A_i and A_j are equivalence classes of R, there must exist elements a and b in A such that $A_i = [a]$ and $A_j = [b]$. By the corollary to theorem 1,

$$\text{either } [a] \cap [b] = \emptyset \text{ or } [a] = [b].$$

But $[a] \neq [b]$ because $A_i \neq A_j$. Hence $[a] \cap [b] = \emptyset$. Thus $A_i \cap A_j = \emptyset$, and so A_i and A_j are disjoint [as was to be shown]. $\qquad\square$

There are m different congruence classes modulo m, corresponding to the m different remainders possible when an integer is divided by m. The m congruence classes are also denoted by $[0]_m, [1]_m, \ldots, [m-1]_m$. They form a partition of the set of integers.

$$[a]_m = \{x \in \mathbb{Z} \mid x \equiv a \pmod{m}\}.$$

Example 4. *What is equivalence class of $1, 2$ for congruence modulo 5? Let $R = \{(a, b) \mid a \equiv b \pmod{5}\}$.*

Solution. For each integer a,

$$
\begin{aligned}
[a] &= \{x \in \mathbb{Z} \mid x \, R \, a\} \\
&= \{x \in \mathbb{Z} \mid 5 \mid (x - a)\} \\
&= \{x \in \mathbb{Z} \mid x - a = 5k, \text{ for some integer } k\}.
\end{aligned}
$$

Therefore,

$$[a] = \{x \in \mathbb{Z} \mid x = 5k + a, \text{ for some integer } k\}.$$

In particular,

$$
\begin{aligned}
[1] &= \{x \in \mathbb{Z} \mid x = 5k + 1, \text{ for some integer } k\} \\
&= \{\ldots, -14, -9, -4, 1, 6, 11, 16, 21, \ldots\}
\end{aligned}
$$

and

$$
\begin{aligned}
[2] &= \{x \in \mathbb{Z} \mid x = 5k + 2, \text{ for some integer } k\} \\
&= \{\ldots, -13, -8, -3, 2, 7, 12, 17, 22, \ldots\}. \qquad \square
\end{aligned}
$$

Example 5. *How many distinct equivalence classes are there modulo 5?*

Solution. There are five distinct equivalence classes, modulo 5: $[0], [1], [2], [3]$, and $[4]$. $\qquad \square$

The last examples above illustrate a very important property of equivalence classes, namely that an equivalence class may have many different names. In the above example, for instance, the class of 0, $[0]$, may also be called the class of 5, $[5]$, or the class of -10, $[-10]$. But what the class is, is the set

$$\{x \in \mathbb{Z} \mid x = 5k, \text{ for some integers } k\}.$$

Definition 5. *Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A representative of the class S is any element a such that $[a] = S$.*

If a is any element of an equivalence class S, then $S = [a]$. Hence every element of an equivalence class is a representative of that class.

Example 6. *Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,*

$$A = \mathbb{Z} \times (\mathbb{Z} - \{0\}).$$

Define a relation R on A as follows: $\forall (a, b), (c, d) \in A$,

$$(a, b) \, R \, (c, d) \Leftrightarrow ad = bc.$$

The fact is that R is an equivalence relation. Describe the distinct equivalence classes of R.

Solution. There is one equivalence class for each distinct rational number. Each equivalence class consists of all ordered pairs (a, b) that, if written as fractions $\frac{a}{b}$, would equal each other. The reason for this is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related. For instance, the class of $(1, 2)$ is

$$[(1,2)] = \{(1,2), (-1,-2), (2,4), (-2,-4), (3,6), (-3,-6), \ldots\}$$

because

$$\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6} = \cdots.$$

It is possible to expand this result to define operations of addition and multiplication on the equivalence classes of R that satisfy all the same properties as the addition and multiplication of rational numbers. It follows that the rational numbers can be defined as equivalence classes of ordered pairs of integers. $\square$